Unit 1

Elementary Probability Theory: Definition of probability; Conditional probability; independent and dependent events; mutually exclusive events; probability distributions; relation between population, sample mean and variance; Combinatorial analysis; Fundamental principle of permutations and combinations; approximation of n!

Experiment, Outcome, and Sample Space

There are only a few things, if any, in life that have definite, certain, and sure outcomes. Most things in life have uncertain outcomes. For example, will you get an A grade in the statistics class that you are taking this semester? Suppose you recently opened a new business; will it be successful? Your friend will be getting married next month. Will she be happily married for the rest of her life? You just bought a lottery ticket. Will it be a winning ticket? You are playing black jack at a casino. Will you be a winner after 20 plays of the game? In all these situations, the outcomes are random. You know the various outcomes for each of these games/statistical experiments, but you do not know which one of these outcomes will actually occur. As another example, suppose you are working as a quality control manager at a factory where they are making hockey pucks. You randomly select a few pucks from the production line and inspect them for being good or defective. This act of inspecting a puck is an example of a statistical **experiment**. The result of this inspection will be that the puck is either "good" or "defective." Each of these two observations is called an **outcome** (also called a **basic** or **final outcome**) of the experiment, and these outcomes taken together constitute the **sample space** for this experiment. The elements of a sample space are called **sample points**.

Experiment, Outcomes, and Sample Space An **experiment** is a process that, when performed, results in one and only one of many observations. These observations are called the **outcomes** of the experiment. The collection of all outcomes for an experiment is called a **sample space**.

Table 4.1 lists some examples of experiments, their outcomes, and their sample spaces.

Table 4.1 Examples of Experiments, Outcomes, and Sample Spaces

Experiment	Outcomes	Sample Space
Toss a coin once	Head, Tail	{Head, Tail}
Roll a die once	1, 2, 3, 4, 5, 6	{1, 2, 3, 4, 5, 6}
Toss a coin twice	*HH*, *HT*, *TH*, *TT*	{*HH*, *HT*, *TH*, *TT*}
Play lottery	Win, Lose	{Win, Lose}
Take a test	Pass, Fail	{Pass, Fail}
Select a worker	Male, Female	{Male, Female}

The sample space for an experiment can also be illustrated by drawing a tree diagram. In a **tree diagram**, each outcome is represented by a branch of the tree. Tree diagrams help us understand probability concepts by presenting them visually. Examples 4–1 through 4–3 describe how to draw the tree diagrams for statistical experiments.

Drawing the tree diagram: one toss of a coin.

EXAMPLE 4–1 One Toss of a Coin

Draw the tree diagram for the experiment of tossing a coin once.

Solution This experiment has two possible outcomes: head and tail. Consequently, the sample space is given by

$$\text{Sample space} = \{H, T\}, \qquad \text{where } H = \text{Head} \quad \text{and} \quad T = \text{Tail}$$

To draw a tree diagram, we draw two branches starting at the same point, one representing the head and the second representing the tail. The two final outcomes are listed at the ends of the branches as shown in Figure 4.1. ■

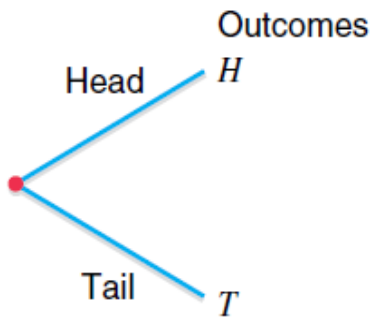

Figure 4.1 Tree diagram for one toss of a coin.

EXAMPLE 4–2 Two Tosses of a Coin

Draw the tree diagram for the experiment of tossing a coin twice.

Drawing the tree diagram: two tosses of a coin.

Solution This experiment can be split into two parts: the first toss and the second toss. Suppose that the first time the coin is tossed, we obtain a head. Then, on the second toss, we can still obtain a head or a tail. This gives us two outcomes: *HH* (head on both tosses) and *HT* (head on the first toss and tail on the second toss). Now suppose that we observe a tail on the first toss. Again, either a head or a tail can occur on the second toss, giving the remaining two outcomes: *TH* (tail on the first toss and head on the second toss) and *TT* (tail on both tosses). Thus, the sample space for two tosses of a coin is

$$\text{Sample space} = \{HH, HT, TH, TT\}$$

The tree diagram is shown in Figure 4.2. This diagram shows the sample space for this experiment. ■

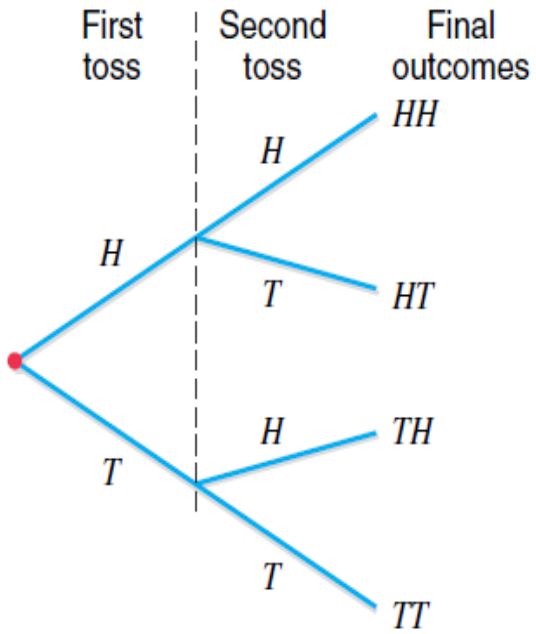

Figure 4.2 Tree diagram for two tosses of a coin.

4.1.1 Simple and Compound Events

An **event** consists of one or more of the outcomes of an experiment.

Event An **event** is a collection of one or more of the outcomes of an experiment.

An event may be a **simple event** or a **compound event**. A simple event is also called an **elementary event**, and a compound event is also called a **composite event**.

Simple Event

Each of the final outcomes for an experiment is called a **simple event**. In other words, a simple event includes one and only one outcome. Usually, simple events are denoted by E_1, E_2, E_3, and so forth. However, we can denote them by any other letter—that is, by A, B, C, and so forth. Many times we denote events by the same letter and use subscripts to distinguish them, as in $A_1, A_2, A_3, \ldots$.

Simple Event An event that includes one and only one of the (final) outcomes for an experiment is called a **simple event** and is usually denoted by E_i.

EXAMPLE 4–4 Selecting Two Workers

s.

Reconsider Example 4–3 on selecting two workers from a company and observing whether the worker selected each time is a man or a woman. Each of the final four outcomes (*MM*, *MW*, *WM*, and *WW*) for this experiment is a simple event. These four events can be denoted by E_1, E_2, E_3, and E_4, respectively. Thus,

$$E_1 = \{MM\}, \quad E_2 = \{MW\}, \quad E_3 = \{WM\}, \quad \text{and} \quad E_4 = \{WW\}$$ ■

Compound Event

A **compound event** consists of more than one outcome.

Compound Event A **compound event** is a collection of more than one outcome for an experiment.

Compound events are denoted by A, B, C, D, . . . , or by A_1, A_2, A_3, . . . , or by B_1, B_2, B_3, . . . , and so forth. Examples 4–5 and 4–6 describe compound events.

Illustrating a compound event: two selections.

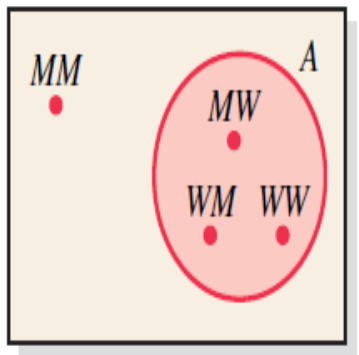

Figure 4.4 Diagram showing event A.

EXAMPLE 4–5 Selecting Two Workers

Reconsider Example 4–3 on selecting two workers from a company and observing whether the worker selected each time is a man or a woman. Let A be the event that at most one man is selected. Is event A a simple or a compound event?

Solution Here *at most one man* means one or no man is selected. Thus, event A will occur if either no man or one man is selected. Hence, the event A is given by

$$A = \text{at most one man is selected} = \{MW, WM, WW\}$$

Because event A contains more than one outcome, it is a compound event. The diagram in Figure 4.4 gives a graphic presentation of compound event A. ■

Probability **Probability** is a numerical measure of the likelihood that a specific event will occur.

4.2.1 Two Properties of Probability

There are two important properties of probability that we should always remember. These properties are mentioned below.

1. The probability of an event always lies in the range 0 to 1.

Whether it is a simple or a compound event, the probability of an event is never less than 0 or greater than 1. We can write this property as follows.

First Property of Probability

$$0 \leq P(E_i) \leq 1$$
$$0 \leq P(A) \leq 1$$

An event that cannot occur has zero probability and is called an **impossible (or null) event**. An event that is certain to occur has a probability equal to 1 and is called a **sure (or certain) event**. In the following examples, the first event is an impossible event and the second one is a sure event.

$$P(\text{a tossed coin will stand on its edge}) = 0$$

$$P(\text{a child born today will eventually die}) = 1.0$$

There are very few events in real life that have probability equal to either zero or 1.0. Most of the events in real life have probabilities that are between zero and 1.0. In other words, these probabilities are greater than zero but less than 1.0. A higher probability such as .82 indicates that the event is more likely to occur. On the other hand, an event with a lower probability such as .12 is less likely to occur. Sometime events with very low (.05 or lower) probabilities are also called **rare events**.

2. The sum of the probabilities of all simple events (or final outcomes) for an experiment, denoted by $\sum P(E_i)$, is always 1.

Second Property of Probability For an experiment with outcomes $E_1, E_2, E_3, \ldots,$

$$\sum P(E_i) = P(E_1) + P(E_2) + P(E_3) + \cdots = 1.0$$

For example, if you buy a lottery ticket, you may either win or lose. The probabilities of these two events must add to 1.0, that is:

$$P(\text{you will win}) + P(\text{you will lose}) = 1.0$$

Similarly, for the experiment of one toss of a coin,

$$P(\text{Head}) + P(\text{Tail}) = 1.0$$

4.2.2 Three Conceptual Approaches to Probability

How do we assign probabilities to events? For example, we may say that the probability of obtaining a head in one toss of a coin is .50, or that the probability that a randomly selected family owns a home is .68, or that the Los Angeles Dodgers will win the Major League Baseball championship next year is .14. How do we obtain these probabilities? We will learn the procedures that are used to obtain such probabilities in this section. There are three conceptual approaches to probability: (1) classical probability, (2) the relative frequency concept of probability, and (3) the subjective probability concept. These three concepts are explained next.

Classical Probability

Many times, various outcomes for an experiment may have the same probability of occurrence. Such outcomes are called **equally likely outcomes**. The classical probability rule is applied to compute the probabilities of events for an experiment for which all outcomes are equally likely. For example, head and tail are two equally likely outcomes when a fair coin is tossed once. Each of these two outcomes has the same chance of occurrence.

Equally Likely Outcomes Two or more outcomes that have the same probability of occurrence are said to be **equally likely outcomes**.

Classical Probability Rule to Find Probability Suppose E_i is a simple event and A is a compound event for an experiment with equally likely outcomes. Then, applying the classical approach, the probabilities of E_i and A are:

$$P(E_i) = \frac{1}{\text{Total number of outcomes for the experiment}}$$

$$P(A) = \frac{\text{Number of outcomes favorable to } A}{\text{Total number of outcomes for the experiment}}$$

Examples 4–7 through 4–9 illustrate how probabilities of events are calculated using the classical probability rule.

EXAMPLE 4–7 One Toss of a Coin

Find the probability of obtaining a head and the probability of obtaining a tail for one toss of a coin.

Solution If we toss a fair coin once, the two outcomes, head and tail, are equally likely outcomes. Hence, this is an example of a classical experiment. Therefore,[1]

$$P(\text{head}) = \frac{1}{\text{Total number of outcomes}} = \frac{1}{2} = \mathbf{.50}$$

[1]If the final answer for the probability of an event does not terminate within four decimal places, it will be rounded to four decimal places.

Relative Frequency Concept of Probability

Suppose we want to calculate the following probabilities:

1. The probability that the next car that comes out of an auto factory is a "lemon"
2. The probability that a randomly selected family owns a home
3. The probability that a randomly selected woman is an excellent driver
4. The probability that an 80-year-old person will live for at least 1 more year
5. The probability that a randomly selected adult is in favor of increasing taxes to reduce the national debt
6. The probability that a randomly selected person owns a sport-utility vehicle (SUV)

These probabilities cannot be computed using the classical probability rule because the various outcomes for the corresponding experiments are not equally likely. For example, the next car manufactured at an auto factory may or may not be a lemon. The two outcomes, "the car is a lemon" and "the car is not a lemon," are not equally likely. If they were, then (approximately) half the cars manufactured by this company would be lemons, and this might prove disastrous to the survival of the factory.

Using Relative Frequency as an Approximation of Probability If an experiment is repeated n times and an event A is observed f times where f is the frequency, then, according to the relative frequency concept of probability:

$$P(A) = \frac{f}{n} = \frac{\text{Frequency of } A}{\text{Sample size}}$$

EXAMPLE 4–10 Lemons in Car Production

Approximating probability by relative frequency: sample data.

Ten of the 500 randomly selected cars manufactured at a certain auto factory are found to be lemons. Assuming that lemons are manufactured randomly, what is the probability that the next car manufactured at this auto factory is a lemon?

Solution Let n denote the total number of cars in the sample and f the number of lemons in n. Then, from the given information:

$$n = 500 \quad \text{and} \quad f = 10$$

Using the relative frequency concept of probability, we obtain

$$P(\text{next car is a lemon}) = \frac{f}{n} = \frac{10}{500} = \mathbf{.02}$$

This probability is actually the relative frequency of lemons in 500 cars. Table 4.2 lists the frequency and relative frequency distributions for this example.

Table 4.2 **Frequency and Relative Frequency Distributions for the Sample of Cars**

Car	f	Relative Frequency
Good	490	490/500 = .98
Lemon	10	10/500 = .02
	$n = 500$	Sum = 1.00

The column of relative frequencies in Table 4.2 is used as the column of approximate probabilities. Thus, from the relative frequency column,

Subjective Probability

Many times we face experiments that neither have equally likely outcomes nor can be repeated to generate data. In such cases, we cannot compute the probabilities of events using the classical probability rule or the relative frequency concept. For example, consider the following probabilities of events:

1. The probability that Carol, who is taking a statistics course, will earn an A in the course
2. The probability that the Dow Jones Industrial Average will be higher at the end of the next trading day
3. The probability that the New York Giants will win the Super Bowl next season
4. The probability that Joe will lose the lawsuit he has filed against his landlord

Neither the classical probability rule nor the relative frequency concept of probability can be applied to calculate probabilities for these examples. All these examples belong to experiments that have neither equally likely outcomes nor the potential of being repeated. For example, Carol, who is taking statistics, will take the test (or tests) only once, and based on that she will either earn an A or not. The two events "she will earn an A" and "she will not earn an A" are not equally likely. Also, she cannot take the test (or tests) again and again to calculate the relative frequency of getting or not getting an A grade. She will take the test (or tests) only once. The probability assigned to an event in such cases is called **subjective probability**. It is based on the individual's

Subjective Probability **Subjective probability** is the probability assigned to an event based on subjective judgment, experience, information, and belief. There are no definite rules to assign such probabilities.

Subjective probability is assigned arbitrarily. It is usually influenced by the biases, preferences, and experience of the person assigning the probability.